FURTHER MATHEMATICS

Paper 9231/01
Paper 1

General comments

Many high quality scripts were presented in response to this examination and there were few very poor scripts. Candidates generally took care to make clear the method of solution. However, there were again many elementary errors in the working so that script totals were needlessly depressed. But for these, the overall standard would have been higher.

Few candidates gave evidence of being in time trouble. In fact almost all candidates produced some sensible work in response to at least 8 questions. It was also noteworthy that there were few misreads. However, comment must be made on the large increase in rubric infringements in that a significant minority of candidates handed in responses to both the alternatives of **Question 11**. Since only one of them can be awarded any credit, then of course, such a practice is a serious waste of examination time.

The overall impression gained by the Examiners was that candidates had a detailed knowledge of most of the syllabus. An examination such as this is a test of how well this knowledge can be turned into an effective problem solving capability. In this respect, most topics fared at least quite well. Induction, curve sketching, linear spaces and some aspects of 3-dimensional metric vectors stand out as areas of uncertainty, while summation of series, convergence of infinite series (or not), and applications of the calculus generated a lot of outstanding work.

Finally, it is good to record further improvements in the standard of presentation of work.

Comments on specific questions

Question 1

The majority of candidates produced a complete and correct response to this question. Very few failed to establish the resolution

$$u_n = \frac{1}{4n^2 - 1} = \frac{1}{4n-2} - \frac{1}{4n+2},$$

and to go on to apply the difference method to obtain

$$\frac{1}{2} - \frac{1}{4N+2} \quad (*)$$

At this stage, there were some notational confusion in that n and not N was used in the sum function.

For the rest, some candidates got involved in complicated convergence tests such as might be used when S_N is unobtainable in a simple form. Here, it is sufficient to show that (*) implies $\lim_{N\to\infty} S_N = \frac{1}{2}$, for such a result establishes both the convergence of $\sum_{n=1}^{\infty} u_n$ as well as its value.

Answers: $\frac{1}{2} - \frac{1}{4N+2}$; $\frac{1}{2}$.

Question 2

Most responses showed an approximately correct sketch of the region R with the upper boundary a line $\theta = \frac{\pi}{6}$. However, about half of all candidates failed to produce a correct form for R at the point (where the tangent is perpendicular to the initial line.

For the rest of this question, a correct integral representation of the required area appeared in most scripts. Nevertheless, about a minority of candidates did not, or could not, go on to an entirely correct completion. All that was necessary was to write

$$\frac{1}{2}\int_0^{\pi/6} \cos^2 2\theta \mathrm{d}\theta = \frac{1}{4}\int_0^{\pi/6} (1 + \cos 4\theta)\mathrm{d}\theta$$

and then to carry out the integration process. Nothing beyond the methodology of basic A Level mathematics was needed at this stage of the question.

Finally, it must be remarked that some responses showed a decimal answer even though the question specifically asks for an exact result. This error suggests that some candidates were unaware that an irrational number cannot be represented by a finite decimal and/or did not know that π is an irrational number.

Answer: $\frac{\sqrt{3}}{32} + \frac{\pi}{24}$.

Question 3

The overall impression given by about half of all responses was that of an uninformed methodology. The hope, apparently, was that if enough algebra appeared then the argument would look after itself. However, this was far from being the case.

In the first place, it is helpful to define

$$\phi(n) \equiv 23^{2n} + 31^{2n} + 46 \text{ for all non-negative } n.$$

This simple notational expedient will simplify the argument later on. Thus to begin with the inductive hypothesis H_k can be formulated as:

$$H_k : 48|\phi(k) \text{ some integer } k > 0.$$

After some algebra the result

$$\phi(k + 1) = \phi(k) + 48(11.23^{2k} + 20.31^{2k})$$

is obtained and this shows that $H_k \Rightarrow H_{k+1}$. Completion of the inductive argument then requires the observation that H_0 is true, since $\phi(0) = 1 + 1 + 46 = 48$. However, many arguments were deficient in some or all of the following ways.

- There was no coherent definition of the inductive hypothesis,
- The central algebra only got as far as showing

$$H_k \Rightarrow \phi(k + 1) - \phi(k) = 48(11.23^{2k} + 20.31^{2k}).$$

- The hypothesis H_1 was shown to be correct, but then it was claimed later that $n|\phi(n)$ for all $n > 0$.

Question 4

Responses to the first part of this question were generally complete and correct, but in contrast candidates made any significant progress with the remainder.

Candidates generally established the echelon form

$$\begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

for the matrix **A** and hence obtained the dimension of R_T, the range space of T. Some, however, unnecessarily worked with the transpose of **A** and so increased the risk of introducing errors into the working.

For the last part of this question, suppose that $\mathbf{b_1}$, $\mathbf{b_2}$ are basis vectors of R_T and consider the linear form $\mathbf{L} = \lambda\, \mathbf{Mb_1} + \mu\, \mathbf{Mb_2}$. Since **M** is given to be non-singular, then $\mathbf{M}^{-1}\mathbf{L}$ exists and equals $\lambda\, \mathbf{b_1} + \mu\, \mathbf{b_2}$ which, since $\dim(R_T) = 2$, cannot be the zero vector unless λ and μ are both zero. Thus $\mathbf{Mb_1}$ and $\mathbf{Mb_2}$ are linearly independent and so $\dim(S) = 2$.

Question 5

There was much variation in the quality of responses to this question. About half of all candidates produced correct or nearly correct solutions to all parts, while a minority of candidates made almost no progress beyond part **(i)**.

(i) Almost without exception the vertical asymptote was identified as $x = -1$. In contrast, a significant minority of candidates failed to obtain $y = 2x + 3$ as the diagonal asymptote and instead wrote down $y = 2x$. This particular inaccuracy almost totally undermined any prospect of success with the rest of the question.

(ii) Those candidates who wrote the equation of C in the form $y = 2x + 3 - \dfrac{6}{x+1}$ had little difficulty in determining the required set of x. In contrast, most of those who did not proceed in this way got involved in complicated and/or erroneous inequality arguments. Nevertheless, in one way or another, the majority of candidates did produce correct answers to this part even though, in many cases, there was no valid supporting argument.

(iii) Most sketches were satisfactory in that the asymptotes were placed appropriately relative to the axes and that two branches appeared whose location and intersections with the axes were correct. However, the majority of sketches had some deficiency with regard to the form at infinity. In this case there are four such forms and some care is needed to ensure that each of them relates to its associated asymptote in the right way.

Answers: **(i)** $x = -1$, $y = 2x + 3$; **(ii)** $x < -1$, $x > -1$.

Question 6

Most candidates produced good work in response to this question. In general, the working for part **(a)** was superior to that for part **(b)**. Overall, this question brought to light a lack of understanding of the laws of indices.

(i) This very standard question on the evaluation of arc length was answered accurately by most candidates. A few used a substitution such as $t = 1 + 3x$ to evaluate $\int_0^1 \sqrt{1+3x}\,dx$ which for this level of examination is an unnecessary complication.

(ii) Some candidates differentiated the equation in its given form. They also evaluated $\frac{\mathrm{d}y}{\mathrm{d}x}$ and so immediately obtained the required result. Others first wrote $\frac{\mathrm{d}y}{\mathrm{d}x} = (x^4 - y^3 + 6)^{1/3}$ and carried out the differentiation. However, this strategy was usually undermined by at least one basic error.

Answers: **(a)** $\frac{14}{9}$; **(b)** $-\frac{1}{6}$.

Question 7

About half of all responses showed a complete and correct derivation of the displayed reduction formula. The preferred strategy was an application of the integration by parts rule based on $u = \sin^{n-1}x$, $v = -\cos x$. Although much of the working in this context was essentially correct it was nonetheless badly undermined by the omission of limits in most, if not all, of the integrals involved. In contrast, a small minority started with a consideration of $D(\cos x \sin^{n+1}x)$ and so easily obtained the required result.

Comment must also be made on responses which showed a correct derivation of the result $I_n = \frac{n-1}{n} I_{n-2}$ and then, without explanation, went on to write down the required result.

In the remainder of the question, almost all candidates understood that $\bar{y} = \frac{I_8}{2I_4}$ (**) and were able to relate this basic result to the reduction formula. A minority of candidates saw at once that as $I_8 = \left(\frac{7}{8}\right)\left(\frac{5}{6}\right)I_4$ then nothing beyond simple arithmetic was needed for the evaluation of $\bar{y}$. The majority attempted to evaluate I_8 and I_4 separately and then went on to use their results in (**). However, this slower, more error prone strategy was frequently derailed by elementary errors.

Answer: $\frac{35}{96}$.

Question 8

Differential equations is usually a popular topic in these examinations and for this reason the general quality of responses to this question was not as high as might be expected. Again, comment must be made on the negative effect of many elementary errors in the working.

Most responses began with a correct derivation of the complementary function. However, for the particular integral things did not go so well in that a significant minority of candidates thought this must be of the form $\lambda t e^{-3t}$. Possibly the motivation for this basic error was the appearance of e^{-3t} in the complementary function and this led to the application of a standard procedure without due consideration.

The majority of responses showed a correct use of the initial conditions. However, there were some candidates who applied them to $y = e^{-3t}(A\cos 4t + B\sin 4t)$ as if this was the general solution. Working for the last part was often superficial and proved nothing, or was unnecessarily complicated. In fact it is only necessary to show that $y = e^{-3t}(3\cos 4t + 4\sin 4t + 5)$ can be expressed in the form $e^{3t}y = 5\cos(4t + \phi) + 5$ from which the required result follows immediately.

Answer: $y = e^{-3t}(A\cos 4t + B\sin 4t + 5)$.

Question 9

This was yet another question where the methodology used by most candidates was appropriate, but where, on account of many working errors, complete and correct responses were very much in a minority.

The first two displayed results are standard and almost no candidates failed to es… Subsequently, most responses showed an attempt to use the binomial expansion of $\left(z + \frac{1}{z}\right)$… constructive way. However, not infrequently this function of z was interpreted as $\cos^7\theta$ and not as 128cos… Likewise, it was common for some or all of $z^n + \frac{1}{z^n}$ where n = 1, 3, 5, 7 to be regarded as $\cos n\theta$ and not as $2\cos n\theta$. Such errors, of course, precluded any possibility of obtaining the correct values for p, q, r and s.

Most responses to the final part of the question began with the correct integral representation of the mean value, that is with $\frac{4}{\pi}\int_0^{\pi/4}\cos^7 2\theta \, d\theta$. Later working then made use of the result of the previous part of the question though earlier errors were the main cause of failure at this stage.

Answers: $\frac{1}{64}\cos 7\theta + \frac{7}{64}\cos 5\theta + \frac{21}{64}\cos 3\theta + \frac{35}{64}\cos\theta$; $\frac{32}{35}\pi$.

Question 10

Candidates generally found some difficulty with this question. Many failures occurred in the first part of the question.

Responses generally exhibited $p = 24$ and subsequently stated or implied that the vector $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ is perpendicular to the plane Π. However, about half of all candidates were unable to obtain possible values for q, r, s and t. In this context it was common to see complicated arguments based on the observation that $(q\mathbf{i} + r\mathbf{j}) \times (s\mathbf{i} + t\mathbf{k})$ is parallel to the normal vector $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, but in most cases these were not successful.

The next part of this question again divided candidates into two main categories. On the one hand, there were those who took the most direct route, that is they verified that, for all θ, the given scalar equation of Π is checked out when $x = 29 + 5\theta$, $y = -2 - 6\theta$, $z = -1 + 2\theta$. On the other, there were those who verified at least one of (a) that a particular point of l is in Π, (b) l is perpendicular to $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$. However, not all members of the second category of candidates verified both (a) and (b).

For the last part, strategies were soundly based and the working was generally accurate. There were many correct responses. Frequently these began with $(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) \times (5\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}) = 30\mathbf{i} + 16\mathbf{j} - 27\mathbf{k}$ so indicating that almost all candidates were familiar with the vector product and were able to apply it in a relevant way.

Answers: $\mathbf{r} = 24\mathbf{i} + \lambda(3\mathbf{i} - 2\mathbf{j}) + \mu(2\mathbf{i} - \mathbf{k})$ or equivalent; $30x + 16y - 27z = 865$.

Question 11 EITHER

Almost all responses began with something like $\Sigma\alpha^2 = 9 - 2 \times 5 = -1$ but subsequent working was markedly less convincing.

For the next part full credit could be obtained by arguing along the following lines:

(1) All roots real $\Rightarrow \Sigma\alpha^2 > 0$.

(2) Real coefficients $\Rightarrow$ complex roots occur in conjugate pairs.

(3) Hence as $\Sigma\alpha^2 < 0$ then there are < 2 real roots.

However, the majority of responses missed out (2).

The working for the third section of this question was generally poorly structured. Odd fragments of possibly relevant results were scattered around without cohesion. In the first place it is helpful to define $\phi(x) \equiv x^4 + 3x^3 + 5x^2 + 12x + 4$ so that $\phi(0) = 4$, $\phi(-1) = -5$, $\phi(-3) = 13$. The change of sign of $\phi(x)$ in the interval $[-3, -1]$ implies an odd number of real roots in this interval and likewise for the interval $[-1, 0]$. Since it has been proved that there are no more than 2 real roots, then there must be exactly one real root in each of the intervals $[-3, -1]$ and $[-1, 0]$. Thus there are exactly 2 real roots in the interval $[-3, 0]$.

The basic two components of the proof of the final result are $\alpha\beta\gamma\delta = 4$ and γ, δ complex $\Rightarrow |\gamma| = |\delta|$. The required result can then easily be derived.

Question 11 OR

Only about half of all candidates could establish the introductory result. In fact the proof is simple and depends, in the first place, on the observation that $\mathbf{BM} = \mathbf{MAM}^{-1}\mathbf{M} = \mathbf{MA}$. From there it follows that $\mathbf{B}(\mathbf{Mx}) = (\mathbf{BM})\mathbf{x} = (\mathbf{MA})\mathbf{x} = \ldots = \lambda(\mathbf{Mx})$ which result, by definition, shows **Mx** to be an eigenvector of **B** with λ as the corresponding eigenvalue.

(i) Most responses indicated in some way that the eigenvalues of **A** are 1, –3, –5. A few candidates, however, unnecessarily first established the characteristic equation of **A** and then eventually obtained the eigenvalues. Generally the methodology for producing a set of corresponding eigenvectors was correct, but there were many elementary errors in the working.

(ii) Responses usually showed an application of the first result leading to the conclusion that the eigenvalues of **B** are also 1, –3, –5. They also multiplied their eigenvectors for **A** by the given matrix **M**, but earlier errors carried on to this point to the extent that only a minority of candidates obtained a correct set of eigenvector for **B**.

(iii) The correct methodology appeared in most responses but by this stage relatively few scripts were error free.

Answers: **(i)** 1, –3, –5; $\begin{pmatrix} 24 \\ 6a \\ ac+4b \end{pmatrix}$, $\begin{pmatrix} 0 \\ 2 \\ c \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$; **(ii)** 1, –3, –5; $\begin{pmatrix} ac+4b+24 \\ 6a \\ ac+4b \end{pmatrix}$, $\begin{pmatrix} c \\ 2 \\ c \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$;

(iii) A possible matrix **Q** has the eigenvectors of **B** as columns; $\mathbf{D} = \text{diag}[1, (-3)^n, (-5)^n]$.

FURTHER MATHEMATICS

Paper 9231/02

Paper 2

General comments

The disparity in performance which has often been evident in the past between Mechanics and Statistics was rather less obvious this time, following the trend which was first commented on last year. Indeed, somewhat unusually, the majority of candidates chose to attempt the Mechanics option in the single question offering alternative choices, namely **Question 11**. As usual the great majority of candidates attempted all the questions on the paper, indicating that the time pressure was not unduly great.

Overall there were some good answers to all questions, though as always some seemed to be more challenging than others, as measured by the level of performance. However this variation was not too marked, and on the whole the paper worked well in allowing candidates to demonstrate their grasp of the subject.

Comments on specific questions

Question 1

As is usual in such collisions, the keys to solution are momentum conservation and restitution, in this case parallel and perpendicular to the plane respectively. The resulting two equations may then be readily solved in various alternative ways. The most frequent cause of error was confusion over the directions of the velocity components, though more seriously a few candidates thought that the ratio of the final to the initial speed is equal to the coefficient of restitution.

Answers: 30°; $8/\sqrt{3}$ ms^{-1}.

Question 2

Virtually every candidate used $T = \frac{2\pi}{\omega}$ to determine the value of ω, but many then tried to use some of the other standard SHM equations without success. The key is to appreciate that the deceleration when moving upwards cannot exceed g in magnitude if the particle is not to leave the platform, so that the maximum magnitude of the displacement x and hence the amplitude a follows from $\frac{d^2x}{dt^2} = -\omega^2 x$. Although many candidates applied a standard equation such as $x = a \cos \omega t$ in the final part, this was often unsuccessful since careful thought is needed in relating this to the starting and finishing points of the motion.

Answer: 0.732 s.

Question 3

The two equations resulting from conservation of momentum and from restitution were usually correctly, and the velocities of A and B deduced by solving them. Although consideration of A freq produced the requirement that $e > \frac{7}{8}$, the majority of candidates deduced in a similar way that $e > -\frac{7}{16}$ fo B, overlooking the fact that e cannot be negative and that B always therefore reverses its direction of motion. The method for finding the impulse was known by most though not all candidates, but there were frequently errors over signs when finding the change in velocity of one or other particle as a result of the collision.

Answers: **(i)** $\frac{7-8e}{3}$, $\frac{7+16e}{3}$; **(ii)** $e > \frac{7}{8}$; **(iii)** 0.064 Ns.

Question 4

The required moment of inertia can be found in a variety of ways, though all require application of the standard formula for the moment of inertia of a rectangular lamina and of the parallel axes theorem. One possible approach, for example, is to find the moments of inertia of the original square plate about its Centre and also of the square cut-outs about their centres, and then either combine to give the moment of inertia of the final lamina about its centre and hence about A, or alternatively find the moments of inertia of the plate and the cut-outs about A and then combine. However some candidates went to the opposite extreme and divided the lamina into 14 equal squares, treating each separately before summing the results, while other candidates adopted a middle way approach and considered a small number of square and/or rectangular component pieces of the lamina. One frequently seen misunderstanding was to take M to be the mass of the lamina rather than of the original plate, though this of course meant that the given answer could not be obtained. Another misreading of the question led many candidates to find the angular rather than the linear acceleration, and here too the mass was often incorrect. A more serious error was to apply a familiar formula involving angular velocity rather than acceleration, frequently after rotation through 90°.

Answer: $\frac{168g}{109}$.

Question 5

The two required equations linking friction F and normal reaction R are most easily obtained by vertical and horizontal resolution of forces, and the given inequality is then found by substitution into $F < \mu R$, followed by use of standard trigonometric formulae involving $\frac{1}{2}\alpha$. The inequality was frequently mis-handled however, with common mistakes being to write one or other of the resolution equations as an inequality and to take the friction as equal to μR. Taking moments about A and substituting for R usually produced the given expression for x, but a frequent error in the final part was to obtain $\cos \alpha < \frac{1}{2}$ correctly, but then conclude that $\alpha < 60°$.

Answer: $\alpha > 60°$.

Question 6

Although most candidates realised that $f(t)$ should be integrated in order to find the distribution function $F(t)$, a significant proportion used 0 as the lower limit instead of $\frac{1}{2}$, which prevented the given expression for $G(a)$ being found. The latter follows from showing that it equals $P(T > \frac{40}{a})$ and hence $1 - F(\frac{40}{a})$, though many candidates used unsatisfactory arguments which failed to adequately show the relationship between $G(a)$ and $F(t)$. The final part requires evaluation of $1 - G(60)$, though $G(60)$ was often calculated instead.

Answer: **(ii)** $\frac{7}{16}$.

Question 7

The majority of candidates knew how to find an unbiased estimate of the common variance, thou introduced incorrect factors such as n, $n - 1$ or $\frac{n-1}{n}$. Although there was frequent reference to normal the stated assumption, very few candidates stated sufficiently precisely that both parent populations are assumed to be normal. The confidence interval is found using the tabular value 2.086, but not all candidates used the appropriate formula, and some carried out a test instead of calculating the interval. The latter supports the researcher's claim since it contains 1, but a wide variety of invalid reasons were seen, such as the interval width is about 1 or even that the interval contains $\mu_y - \mu_x$.

Answers: 0.564; [0.64, 1.98].

Question 8

Finding the equation of the regression line rarely presented any difficulty, and most candidates knew that the square of the product moment correlation coefficient equals the product of the gradients in the two equations, though some did not appreciate that the negative square root needs to be chosen. Substitution of 2.25 in place of x in either equation was straightforward, though the fact that x is the independent variable is a better reason for choosing the first equation than simply saying that y can be found more easily from it since there is no real computational advantage in using one equation rather than the other. The closeness of the magnitude of the correlation coefficient to unity is probably the best reason for expecting reliability, though saying that 2.25 lies within the range of tabular values also has some merit.

Answers: **(i)** $y = 9.07 - 2.04x$; **(ii)** -0.93; **(iii)** 4.48 or 4.55.

Question 9

While a reasonable number of candidates derived the equation $1 - e^{-\lambda} = 2e^{-2\lambda}$, not all were able to solve it for λ and hence find the mean from $\frac{1}{\lambda}$. A common fault was to take, in effect, $\ln(e^a + e^b)$ to be $a + b$. In the final part of the question $(1 - p)^5$ needs to be evaluated with $p = \frac{1}{4}$ to find $P(N > 5)$

Answers: $\frac{1}{\ln 2}$; $\frac{243}{1024}$.

Question 10

An appropriate assumption is that of the independence of the faulty batteries, or something similar. In order to calculate the table of expected values of faulty batteries per pack to be compared with the observed data, it is necessary to use the probability $p = \frac{36}{180} = 0.2$ of a battery being faulty, though many candidates used a variety of incorrect values such as $\frac{36}{60}$. Since one of the expected values is less than 5, the last two cells should be combined before calculating χ^2. Comparison of its value, 2.11, with the critical value 2.706 leads to the conclusion that the binomial distribution fits the data. The final part requires use of a binomial expansion with parameter p^3.

Question 11 EITHER

The velocity v at the point where the string slackens may be related to u by conservation of energ follows from a radial resolution of forces since the tension becomes zero. Some candidates were aw this general approach but considered an angle $\frac{\pi}{3}$ with the downward vertical instead of the upward on Showing that, in effect, the particle passes through A presented more problems. It can be accomplished by either considering horizontal and vertical motion separately or applying the trajectory equation, noting that the initial velocity is v at an upward angle of $\frac{\pi}{3}$ to the horizontal. However many candidates mistakenly tried to use similar equations to those in the first part, bringing in the tension of the string even though it is zero while the string is slack. While the speed just before reaching A can be found by using equations of motion, it is much simpler to note that it is necessarily u due to conservation of energy.

Answers: **(i)** $\sqrt{\frac{7ga}{2}}$; **(iii)** u.

Question 11 OR

By forming tables of the observed and expected numbers of male and female candidates who are afraid or not afraid, an expression for χ^2 may be found in the form of a multiple of $(n - 10)^2$. Setting this to be not less than the critical value 3.841 gives the two limiting values 7.1 and 12.9, which ideally should then be used to determine the allowable integer values of n. In the second part of the question, the expected value for frightened female candidates is found to be less than 5, suggesting that cells should be combined, but this produces the obvious difficulty of no degrees of freedom and so this type of test is inadvisable.

Answer: $1 < n < 7$ and $13 < n < 16$.